Clarification: Let z_n a sequence of complex numbers with $z_n \neq -1$. Then $\prod_{n\in\mathbb{N}} (1+z_n)$ converges $\iff$ the sequence

$$P_N := \prod_{n=1}^{N} (1+z_n) \text{ converges.}$$

- $\prod_{n\in\mathbb{N}} (1+z_n)$ converges $\iff \sum_{n\in\mathbb{N}} \log(1+z_n)$ converges.
- $\prod_{n\in\mathbb{N}} (1+z_n)$ converges absolutely

$$\iff \sum_{n\in\mathbb{N}} \log(1+z_n) \text{ converges absolutely}$$

$$\iff \sum_{n\in\mathbb{N}} z_n \text{ converges absolutely}$$

$$\iff \prod_{n\in\mathbb{N}} (1+|z_n|) \text{ converges.}$$

Characters of finite abelian groups

Definition: If G is a finite group, then
$$\hat{G} = \{\chi : G \longrightarrow \mathbb{C}^\times \text{ homomorphism}\}$$
is called the dual group of G. (unitary dual of G).

Lemma: Define $\cdot : \hat{G} \times \hat{G} \longrightarrow \hat{G}$ by $(\chi \cdot \psi)(g) := \chi(g)\psi(g)$, for $g \in G$. Then $(\hat{G}, \cdot)$ abelian group with neutral element $\chi_0 : g \longmapsto 1$.

For any $\chi \in \hat{G}$, the character $\overline{\chi}(g) := \overline{\chi(g)}$ $(g \in G)$ is its multiplicative inverse.

Proof: exercise.

Proposition: Let G finite abelian group.
Then $G \cong \hat{G}$.

Proof: By structure theorem of finite abelian groups, we know $G \cong \prod_{i=1}^{k} (\mathbb{Z}/n_i\mathbb{Z})$,
for some $n_1, \dots, n_k \in \mathbb{N}$.

Therefore there exist $g_1, \dots, g_k \in G$ with $\operatorname{ord}(g_i) = n_i$ s.t any $g \in G$ has unique writing

$$g = g_1^{r_1} \cdots g_k^{r_k}, \quad 1 \le r_i \le n_i.$$

Let $\chi \in \hat{G}$. Then $\chi(g) = \chi(g_1)^{r_1} \cdots \chi(g_k)^{r_k}$.

(enough to determine values of χ on $g_1, \dots, g_k$).

Since g_i has order n_i, then $\chi(g_i)^{n_i} = \chi(g_i^{n_i}) = \chi(e) = 1$

$\Rightarrow \exists\ 1 \le a_i \le n_i$ s.t. $\chi(g_i) = e^{\frac{2\pi i a_i}{n_i}}$.

($\chi(g_i)$ is n_i-th root of unity)

Note that each choice of $1 \le a_i \le n_i$, for $1 \le i \le k$ uniquely determines a character, hence $|G| = |\hat{G}| = \prod_{i=1}^{k} n_i$.

It remains to show G and $\hat{G}$ isomorphic. For $1 \le i \le k$, define the character $\chi_i \in \hat{G}$ s.t.

$$\chi_i(g_i) = e^{\frac{2\pi i}{n_i}}$$
$$\chi_i(g_j) = 1, \text{ for } i \ne j.$$

We see that $\chi_1, \dots, \chi_k$ generate $\hat{G}$ and each $\chi \in \hat{G}$ can be written uniquely as

$$\chi = \chi_1^{r_1} \cdots \chi_k^{r_k}, \quad 1 \le r_i \le n_i.$$

Hence the map $\varphi : G \to \hat{G}$ is an isomorphism
$g_i \mapsto \chi_i$ of groups. $\square$

Remark: The isomorphism is not canonical, it depends on choice of generators $g_1, \dots, g_k$.

Corollary ($\hat{G}$ separates points)
Let G finite abelian. If $g \in G \setminus \{e\}$, there exists $\chi \in \hat{G}$ such that $\chi(g) \neq 1$.

Proof: Let $g \in G \setminus \{e\}$ and $H = \langle g \rangle$, so $|H| > 1$. Suppose for contradiction $H < \ker \chi$, $\forall \chi \in \hat{G}$. Hence we can construct injective map

$$Q : \hat{G} \longrightarrow \widehat{G/H}$$

given by $Q(\chi)(gH) := \chi(g)$, for $g \in G$.
(well defined since $\chi(h) = 1$, $\forall h \in H$, $\forall \chi \in \hat{G}$).
Thus $|G| = |\hat{G}| \leq |\widehat{G/H}| = |G/H| = \frac{|G|}{|H|} < |G|$, contradiction. $\square$

Remark: By above $\hat{\hat{G}} \cong \hat{G} \cong G$. Here isomorphism $\Xi : G \xrightarrow{\sim} \hat{\hat{G}}$ is canonical, given by

$$\Xi(g)(\chi) := \chi(g) \quad (\chi \in \hat{G}, g \in G).$$

Indeed, Ξ is a homomorphism and injective since $\hat{G}$ separates points in G.

$(\Xi(g)(\chi)=1, \forall \chi \in \hat{G} \Leftrightarrow \chi(g)=1, \forall \chi \in \hat{G} \Leftrightarrow g=e).$

Notation: We denote $\chi_0 \in \hat{G}$ to be the identity character $\chi_0(g)=1, \forall g \in G$.

Theorem (Orthogonality relations)

Let G be a finite abelian group.

For all $g \in G$,
$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} |G|, & \text{if } g=e, \\ 0, & \text{if } g \neq e; \end{cases}$$

and for $\chi \in \hat{G}$,
$$\sum_{g \in G} \chi(g) = \begin{cases} |G|, & \text{if } \chi=\chi_0 \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

Proof: We start with second assertion.

Clear when $\chi=\chi_0$.

Suppose $\chi \neq \chi_0$. Hence $\exists h \in G$ s.t. $\chi(h) \neq 1$.

Then
$$\sum_{g \in G} \chi(g) \underset{\substack{\uparrow \\ g \mapsto gh \text{ bijection.}}}{=} \sum_{g \in G} \chi(gh) = \chi(h) \sum_{g \in G} \chi(g)$$

Since $\chi(h) \neq 1 \Longrightarrow \sum_{g\in G} \chi(g) = 0$.

For the first assertion, note that

$$\sum_{\chi\in\hat{G}} \chi(g) = \sum_{\chi\in\hat{G}} \Xi(g)(\chi)$$

and thus it follows from case already proven $\boxtimes$

[Alternatively for first assertion, if $g \neq e$, $\exists\, \chi' \in \hat{G}$ such that $\chi'(g) \neq 1$, hence

$$\sum_{\chi\in\hat{G}} \chi(g) = \sum_{\chi\in\hat{G}} (\chi'\chi)(g) = \chi'(g) \sum_{\chi\in\hat{G}} \chi(g)$$

$$\Longrightarrow \sum_{\chi\in\hat{G}} \chi(g) = 0$$
]

In number theory, we encounter two types of characters: additive characters and multiplicative characters

<u>Additive characters</u>: Let $q \in \mathbb{N}$.

Character of additive group $\psi : (\mathbb{Z}/q\mathbb{Z}, +) \to \mathbb{C}$.

Can view as function $\psi(n) = \psi(n \bmod q)$, $n \in \mathbb{N}$.

There are q additive characters mod q, and since $(\mathbb{Z}/q\mathbb{Z}, +)$ cyclic, they can be uniquely written as

$$\psi(n) = e^{\frac{2\pi i a n}{q}}, \text{ for } 1 \leq a \leq q.$$

Additive character: $\psi(m+n) = \psi(m)\psi(n)$.

Orthogonality relations $\displaystyle\sum_{a \bmod q} e^{\frac{2\pi i a n}{q}} = \begin{cases} q, & \text{if } q \mid n \\ 0, & \text{if } q \nmid n \end{cases}$.

Multiplicative characters: A multiplicative character modulo q is a character of the multiplicative group $((\mathbb{Z}/q\mathbb{Z})^\times, \cdot)$.

Since $|(\mathbb{Z}/q\mathbb{Z})^\times| = \varphi(q)$, there are $\varphi(q)$ mult characters modulo q. Contrary to additive case, in general they cannot be written in explicit form.

$$\chi \in \widehat{(\mathbb{Z}/q\mathbb{Z})^\times} \Longrightarrow \chi(mn) = \chi(m)\chi(n),$$
$$\forall m, n \text{ with } (m,q) = (n,q) = 1.$$

Note that $\chi \in \widehat{(\mathbb{Z}/q\mathbb{Z})^\times}$ can be viewed as a function on classes $(n \bmod q)$, where $(n,q)=1$. Can extend this to all $\mathbb{N}$ as follows:

Definition: (Dirichlet characters).

A map $\chi \in \mathcal{A}$ is a Dirichlet character (mod q) if there exists a character (also) χ on $(\mathbb{Z}/q\mathbb{Z})^\times$ such that

$$\chi(n) = \begin{cases} \chi(n \bmod q), & \text{if } (n,q)=1, \\ 0, & \text{else.} \end{cases}$$

Note: we utilise same symbol for both functions, usually it's clear from context which one is meant.

- Dirichlet characters are completely multiplicative!

Definition (Principal Dirichlet character)

The Dirichlet character corresponding to trivial character χ_0 mod q is called the principal character mod q.

It is given explicitly by $\chi_0(n) := \begin{cases} 1, & (n,q)=1 \\ 0, & \text{otherwise.} \end{cases}$

We can write orthogonality relations as

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q), & \text{if } n \equiv 1 \pmod q \\ 0, & \text{otherwise} \end{cases}$$

and $$\sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q), & \text{if } \chi = \chi_0 \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

We can use Dirichlet characters to look at arithmetic functions in arithmetic progressions:

Corollary (Usefulness of Dirichlet characters)
Let $f \in \mathcal{A}$ an arithmetic function and $a \in \mathbb{Z}$ with $(a,q)=1$.
Then
$$\sum_{\substack{n \leq X \\ n \equiv a(q)}} f(n) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{n \leq X} f(n)\chi(n)$$

Proof: $$\sum_{\substack{n \leq X \\ n \equiv a(q)}} f(n) \underset{\bar{a}a \equiv 1(q)}{=} \sum_{\substack{n \leq X \\ \bar{a}n \equiv 1(q)}} f(n)$$

$$= \sum_{n \leq X} \left(\frac{f(n)}{\varphi(q)} \sum_{\chi \bmod q} \chi(\bar{a}n) \right)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(\bar{a}) \sum_{n \leq X} f(n)\chi(n). \qquad \square.$$

Dirichlet L-functions

Def: Let χ mod q a Dirichlet character. We define the Dirichlet series associated to χ

$$L(s,\chi) := L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This is absolutely convergent for $\operatorname{Re}(s) > 1$
(and hence defines holomorphic function in this region).

χ completely multiplicative $\Rightarrow$ for $\operatorname{Re}(s) > 1$, we have Euler product $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

In particular for $\chi = \chi_0$,

$$L(s,\chi_0) = \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$